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A study of anti-fuzzy bi-ideals in ordered semigroups

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Abstract. In mathematics, an ordered semigroup is a semigroup together with a partial order which is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, design of fast adders and error-correcting codes. A theory of anti-fuzzy bi-ideals in ordered semigroups can be developed. Here we define anti-fuzzy bi-ideals of ordered semigroups and give the main theorem which characterizes bi-ideals of ordered semigroups in terms of anti-fuzzy bi-ideals. Then we characterize left and right simple, completely regular and strongly regular ordered semigroups using anti-fuzzy bi-ideals. We also study the characterization of regular and both regular and intra-regular ordered semigroups in terms of anti-fuzzy bi-ideals.

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1. Introduction

Biswas introduced the concept of an anti-fuzzy subgroup of a group in [2] and studied the basic properties of a group in terms of anti-fuzzy subgroups. Hong and Jun [4] modified Biswas idea and applied it into BCK-algebra. Akram and Dar defined anti-fuzzy left h-ideals of a hemiring and discussed the basic properties (see [1]). Recently Shabir and Nawaz studied anti fuzzy ideals of a semigroup and gave some interesting properties (see [16]). Mordeson et al. in [15] presented an up to date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. The book concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. Basic results on fuzzy subsets, semigroups, codes, finite state machines, and languages are reviewed and introduced, as well as certain fuzzy ideals of a semigroup and advanced characterizations and properties of fuzzy semigroups. With this objective in view, Kuroki [14] introduced the notion of fuzzy bi-ideals in semigroups. The concept of a fuzzy filter in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [11], where some basic properties of fuzzy filters and prime fuzzy ideals were discussed. Fuzzy sets in ordered semigroups were first considered by Kehayopulu and Tsingelis in [11]. The concept of a bi-ideal B of a semigroup S was introduced by Good and Hughes in [3], which is a subsemigroup of S having the property $BSB \subseteq B$. In a similar way bi-ideals in ordered semigroup S is defined to be a subsemigroup of S having the properties: (1) $BSB \subseteq B$ and (2) if $a \in B$ and $S \ni b \leq a$, then $b \in B$. Kehayupolo and Tsingelis [10] studied the relation of bi-ideals and fuzzy bi-ideals of an ordered semigroup.

In this paper, we first give the main theorem which characterizes the bi-ideals of ordered semigroups in terms of their anti-fuzzy bi-ideals. For this, we prove that a non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if the fuzzy subset f_A defined by

$$f_A : S \to \{r, t\}, x \longmapsto \begin{cases} r \text{ if } x \in A \\ t \text{ if } x \notin A \end{cases}$$

where $r, t \in (0, 1)$ and $t \geq r$, is an anti-fuzzy bi-ideal of S. We also prove that a fuzzy subset of an ordered semigroup S is an anti-fuzzy bi-ideal of S if and only if the anti level cut of f defined by $L(f;t) := \{x \in S | f(x) \leq t\}$ is a bi-ideal of S. Then we prove that an ordered semigroup S is *regular*, *left* and *right simple* if and only if every anti-fuzzy bi-ideal of S is a constant mapping. We prove that an ordered semigroup S is completely regular if and only if for each anti-fuzzy bi-ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$. We also discuss regular and both regular and intra-regular ordered semigroups in terms of anti-fuzzy bi-ideals.

2. Preliminaries

In this section we give the basic results and definitions which are essential for the subsequent sections.

An ordered semigroup $(S, \cdot, \leq)$ is an ordered set $(S, \leq)$ at the same time a semigroup $(S, \cdot)$ such that

$$a \leq b \implies xa \leq xb \text{ and } ax \leq bx \text{ for all } a, b, x \in S.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $A \subseteq S$, we denote by

$(A] := \{t \in S | t \leq h \text{ for some } h \in A\}$. For $A, B \subseteq S$, we write, $AB := \{ab | a \in A$ and $b \in B\}$. If $A, B \subseteq S$ then, we have $A \subseteq (A]$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$ and for $A \subseteq B$, we have $(A] \subseteq (B]$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of S is called a left (resp. right) ideal of S (see [12]) if (i) $SA \subseteq A$ (resp. $AS \subseteq S$) and (ii) If $a \in A$, $S \ni b \leq a$ then $b \in A$. A is called a *two-sided ideal* or simply an *ideal* of S if A is both a left and a right ideal of S.

A non empty subset A of an ordered semigroup S is called a *bi-ideal* of S if (i) $A^2 \subseteq A$, (ii) $ASA \subseteq A$ and (iii) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. An ordered semigroup S is called *left* (resp. *right*) *simple* if for every left (resp. right) ideal A of S we have $A = S$. S is called *simple* if it is both left and right simple (see [6]). An ordered semigroup $(S, \cdot, \leq)$ is called *regular* (see [8]) if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$, or equivalently (i) $a \in (aSa]\ \forall\ a \in S$ and (ii) $A \subseteq (ASA]\ \forall\ A \subseteq S$. An ordered semigroup $(S, \cdot, \leq)$ is called *left* (resp. *right) regular* (see [6]) if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$), or equivalently (i) $a \in (Sa^2]$ (resp. $a \in (a^2S]$) $\forall\ a \in S$ and (ii) $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) $\forall\ A \subseteq S$. An ordered semigroup $(S, \cdot, \leq)$ is called *completely regular* (see [6, 7]) if it is regular, left regular and right regular. An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if for every $a \in S$ there exists $x, y \in S$ such that $a = xa^2y$.

Now we recall some fuzzy logic concepts. By a fuzzy subset f of S we mean a mapping from a universe to a unit interval $[0, 1]$ of real numbers.

A fuzzy subset f of a semigroup S is called a *fuzzy subsemigroup* of S (see [10]) if

$$(\forall x, y \in S)(f(xy) \geq \min\{f(x), f(y)\}.$$

A fuzzy subset f of S is called a fuzzy bi-ideal of S (see [10]) if

(i) $(\forall x, y \in S)\ (f(xy) \geq \min\{f(x), f(y)\}),$
(ii) $(\forall x, y, z \in S)(f(xyz) \geq \min\{f(x), f(z)\}),$
(iii) $(\forall x, y \in S)(x \leq y \implies f(x) \geq f(y)).$

3. Anti-fuzzy bi-ideals

In what follows let S denote an ordered semigroup unless otherwise specified.

Definition 3.1. [17] A fuzzy subset f of S is called an *anti-fuzzy subsemigroup* of S if

$$(\forall x, y \in S)(f(xy) \leq \max\{f(x), f(y)\}).$$

Definition 3.2. A fuzzy subset f of S is called an *anti-fuzzy bi-ideal* of S if

(1) $(\forall x, y \in S)(f(xy) \leq \max\{f(x), f(y)\}).$
(2) $(\forall x, y, z \in S)(f(xyz) \leq \max\{f(x), f(z)\}).$
(3) $(\forall x, y \in S)(x \leq y \Rightarrow f(x) \leq f(y)).$

For $a \in S$ we define $A_a = \{(y, z) \in S \times S | a \leq yz\}$ (see [13]).

For any two fuzzy subsets f, g of S the product $f * g$ is defined by (see [17])

$$f * g : S \longrightarrow [0, 1] | a \longmapsto \begin{cases} \bigwedge\limits_{(y,z) \in A_a} \{f(y) \vee g(z)\} & \text{if } A_a \neq \varnothing \\ 1 & \text{if } A_a = \varnothing \end{cases}$$

Let f and g be fuzzy subsets of S then $f \preceq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$.

For an ordered semigroup S, the fuzzy sets "$\mathcal{O}$" and "1" are defined as follows: $\mathcal{O} : S \longrightarrow [0, 1] | x \longmapsto \mathcal{O}(x) = 0$, and $1 : S \longrightarrow [0, 1] | x \longrightarrow 1(x) = 1$.

Proposition 3.3. *If S is an ordered semigroup and f_1, f_2, g_1, g_2 are fuzzy subsets of S such that $f_1 \preceq g_1$ and $f_2 \preceq g_2$ then $f_1 * f_2 \preceq g_1 * g_2$.*

Proof. Let $a \in S$. If $A_a = \varnothing$ then

$$(f_1 * f_2)(a) = 1 \text{ and } (g_1 * g_2)(a) = 1.$$

So

$$(f_1 * f_2)(a) \leq (g_1 * g_2)(a)$$

If $A_a \neq \varnothing$ then

$$\begin{aligned} (f_1 * f_2)(a) &= \bigwedge_{(y,z)\in A_a} \{f_1(y) \vee f_2(z)\} \\ &\leq \bigwedge_{(y,z)\in A_a} \{g_1(y) \vee g_2(z)\}.(\because f_1 \preceq g_1 \text{ and } f_2 \preceq g_2) \\ &= (g_1 * g_2)(a). \end{aligned}$$

□

Proposition 3.4. *Let S be an ordered semigroup and A, $B \subseteq S$ then*

(1) $A \subseteq B$ *if and only if* $f_{B^C} \preceq f_{A^C}$.
(2) $f_{A^C} \vee f_{B^C} = f_{A^C \cup B^C} = f_{(A\cap B)^c}$
(3) $f_{A^C} * f_{B^C} = f_{(AB]^C}$

Proof. (1) and (2) are obvious.

(3) Let $x \in (AB]^C$ then $f_{(AB]^C}(x) = 1$. Since $x \in (AB]^C \implies x \notin (AB]$. If $A_x = \varnothing$ then $(f_{A^C} * f_{B^C})(x) = 1$ and so $(f_{A^C} * f_{B^C})(x) = f_{(AB]^C}(x)$. If $A_x \neq \varnothing$ then

$$(f_{A^C} * f_{B^C})(x) = \bigwedge_{(y,z)\in A_x} \{f_{A^C}(y) \vee f_{B^C}(z)\}.$$

We prove that $\{f_{A^C}(y) \vee f_{B^C}(z)\} = 1$ for all $(y, z) \in A_x$. Now $x \leq yz$ if $y \in A$ and $z \in B$ then $yz \in AB$ and so $x \in (AB]$, which is not possible, so either $y \notin A$ or $z \notin B$. If $y \notin A$ then $f_{A^C}(y) = 1$ and since $f_{B^C}(z) \leq 1$ so we have $\{f_{A^C}(y) \vee f_{B^C}(z)\} = 1$. Similarly if $z \notin B$ then by same argument $\{f_{A^C}(y) \vee f_{B^C}(z)\} = 1$. Hence $f_{A^C} * f_{B^C} = f_{(AB]^C}$ for all $x \in (AB]^C$.

Now let $x \notin (AB]^C \implies f_{(AB]^C}(x) = 0$. Since $x \notin (AB]^C \implies x \in (AB]$ so $x \leq ab$ for some $a \in A$ and $b \in B$, hence $A_x \neq \varnothing$. Thus

$$\begin{aligned} (f_{A^C} * f_{B^C})(x) &= \bigwedge_{(p,q)\in A_x} \{f_{A^C}(p) \vee f_{B^C}(q)\} \\ &\leq f_{A^C}(a) \vee f_{B^C}(b) \end{aligned}$$

Since $a \in A$ and $b \in B$, that is $a \notin A^C$ and $b \notin B^C$ so $f_{A^C}(a) = 0$ and $f_{B^C}(b) = 0 \implies (f_{A^C} * f_{B^C})(x) \leq 0$ which yields $(f_{A^C} * f_{B^C})(x) = 0$. Hence $f_{A^C} * f_{B^C} = f_{(AB]^C}$ for all $x \in (AB]$ and thus $f_{A^C} * f_{B^C} = f_{(AB]^C}$. □

Lemma 3.5. *A non-empty subset A of S is a bi-ideal of S if and only if the fuzzy subset f_A of S defined by*

$$f_A : S \longrightarrow \{r, t\} \;\; x \longmapsto \begin{cases} r \text{ if } x \in A \\ t \text{ if } x \notin A \end{cases}$$

where $r, t \in [0,1]$ and $t \geq r$ is an anti-fuzzy bi-ideal of S.

Proof. Suppose that A is a bi-ideal of S let $x, y \in A$ then $xy \in A$.

(1) If $x, y \in A$ then$f_A(x) = f_A(y) = r$ and so

$$f_A(xy) \leq \max\{f_A(x), f_A(y)\}.$$

(2) If $x \in A$, $y \notin A$ then $f_A(x) = r$, $f_A(y) = t$ and hence

$$f_A(xy) = t \leq \max\{f_A(x), f_A(y)\}$$

(3) If $x \notin A$, $y \in A$ then $f_A(x) = t$ and $f_A(y) = r$ then

$$f_A(xy) = t \leq \max\{f_A(x), f_A(y)\}$$

(4) If $x \notin A$, $y \notin A$ then $f_A(x) = t$ and $f_A(y) = t$ then we have

$$f_A(xy) = t \leq \max\{f_A(x), f_A(y)\} \text{ for all } x, y \in S.$$

Now

(1) Let $x, z \in A, y \in S$ then $xyz \in A$ and hence

$$f_A(xyz) \leq \max\{f_A(x), f_A(z)\} \text{ for all } x, y, z \in S.$$

(2) If $x \notin A$, $z \in A$ then

$$f_A(x) = t \text{ and } f_A(z) = r \implies f_A(xyz) \leq \max\{f_A(x), f_A(z)\}$$

(3) If $x \in A$, $z \notin A$ then $f_A(x) = r$, $f_A(z) = t$, so again

$$f_A(xyz) \leq \max\{f_A(x), f_A(z)\}$$

(4) If $x \notin A$, $z \notin A$ then $f_A(x) = t$, $f_A(z) = t$. Hence

$$f_A(xyz) \leq \max\{f_A(x), f_A(z)\} \text{ for all } x, y, z \in S.$$

Let $S \ni x \leq y \in A$ then $x \in A$ as A is a bi-ideal of S and we have $f_A(x) = r = f_A(y)$. Hence f_A is anti-fuzzy bi-ideal of S.

Conversely, assume that f_A is an anti-fuzzy bi-ideal of S. Let $x, y \in A$ then $f_A(x) = r$ and $f_A(y) = r$. Since f_A is an anti-fuzzy bi-ideal of S, we have

$$f_A(xy) \leq \max\{f_A(x), f_A(y) = \max\{r, r\} = r,$$

and $xy \in A$. Also

$$f_A(xyz) \leq \max\{f_A(x), f_A(z)\} = \max\{r, r\} = r,$$

and hence $xyz \in A$. Let $x, y \in S$ be such that $x \leq y \in A$. Since f_A is an anti-fuzzy bi-ideal of S and $x \leq y$, we have

$$\begin{aligned} f_A(x) &\leq f_A(y) = r \\ &\implies f_A(x) = r, \end{aligned}$$

and hence $x \in A$. □

Remark 3.6. From above Lemma we conclude that a non-empty subset A of S is a bi-ideal of S if and only if the characteristic function of the complement of A, that is f_{A^C} is an anti-fuzzy bi-ideal of S.

Now we characterize anti-fuzzy bi-ideal in terms of its level bi-ideals.

Definition 3.7. Let f be a fuzzy subset of S, then $L(f;t) := \{x \in S | f(x) \leq t\}$ where $t \in [0,1]$ is called the *lower level cut* of f.

Lemma 3.8. *A fuzzy subset f of S is an anti-fuzzy bi-ideal of S if and only if the set $L(f;t) := \{x \in S | f(x) \leq t\}$ where $t \in (0,1]$ is a bi-ideal of S.*

Proof. Suppose that f is an anti-fuzzy bi-ideal of S. Let $x, y \in L(f;t)$ then $f(x) \leq t$ and $f(y) \leq t$. Since f is anti-fuzzy bi-ideal of S, we have

$$f(xy) \leq \max\{f(x), f(y) \leq \max\{t,t\} = t.$$

and so $xy \in L(f;t)$. Again let $x, z \in L(f;t)$ and $y \in S$ then $f(x) \leq t$ and $f(z) \leq t$. Now $f(xyz) \leq \max\{f(x), f(z)\}$ gives $f(xyz) \leq \max\{t,t\} = t$. Hence we get $xyz \in L(f;t)$. Let $x, y \in S$ be such that $x \leq y \in L(f;t)$ then $f(y) \leq t$. Since $x \leq y \Longrightarrow f(x) \leq f(y)$ hence $f(x) \leq t$ and so $x \in L(f;t)$.

Conversely, assume that $L(f;t)(\neq \varnothing)$ is a bi-ideal of S for all $t \in (0,1]$. If there exist $x, y \in S$ such that

$$f(xy) > \max\{f(x), f(y)\} = t,$$

then $t \in (0,1]$, $x, y \in L(f;t)$ but $xy \notin L(f;t)$, a contradiction. Hence

$$f(xy) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S.$$

If there exist $x, y, z \in S$ such that

$$f(xyz) > \max\{f(x), f(z)\} = t_\circ,$$

then $t_\circ \in (0,1]$, $x, z \in L(f;t_\circ)$ but $xyz \notin L(f;t_\circ)$, a contradiction. Hence

$$f(xyz) \leq \max\{f(x), f(z)\} \text{ for all } x, y, z \in S.$$

If there exist $x, y \in S, x \leq y$ such that

$$f(x) > t \geq f(y).$$

Then $y \in L(f;t)$ but $x \notin L(f;t)$, a contradiction. Hence $f(x) \leq f(y)$ for all $x, y \in S$ with $x \leq y$. □

Example 3.9. Let $(S, \cdot, \leq)$ be an ordered semigroup defined as follows

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$\leq := \{(a,a), (a,c), (a,d), (a,e), (b,b), (b,d), (b,e), (c,c), (c,e), (d,d), (d,e), (e,e)\}$.

Define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows:

$$f(a) = 0.5, \quad f(b) = f(c) = f(d) = f(e) = 0.9,$$

then

$$L(f;t) := \begin{cases} \varnothing & \text{if } 0 < t < 0.5 \\ \{a\} & \text{if } 0.5 \leq t < 0.9 \\ \{a,b,c,d,e\} & \text{if } 0.9 \leq t \leq 1.0 \end{cases}$$

and by Lemma 3.8, $L(f;t)$ is a bi-ideal of S for all $t \in (0,1]$ so by Lemma 3.8, f is an anti-fuzzy bi-ideal of S.

Proposition 3.10. *Let S be an ordered semigroup, f an anti-fuzzy bi-ideal of S then we have $f \preceq f * \mathcal{O} * f$.*

Proof. Let $a \in S$. If $A_a = \varnothing$ then $(f * \mathcal{O} * f)(a) = 1 \geq f(a)$. If $A_a \neq \varnothing$ then

$$\begin{aligned}(f * \mathcal{O} * f)(a) &= \bigwedge_{(y,z)\in A_a} \{(f * \mathcal{O})(y) \vee f(z)\} \\ &= \bigwedge_{(y,z)\in A_a} \{\{\bigwedge_{(p,q)\in A_y} \{f(p) \vee \mathcal{O}(q)\}\} \vee f(z)\} \\ &= \bigwedge_{(y,z)\in A_a} \bigwedge_{(p,q)\in A_y} \{f(p) \vee \mathcal{O}(q) \vee f(z)\} \\ &= \bigwedge_{(y,z)\in A_a} \bigwedge_{(p,q)\in A_y} \{f(p) \vee f(z)\}\end{aligned}$$

As $a \leq yz$ and $y \leq pq$ so $a \leq yz \leq pqz$ and since f is an anti-fuzzy bi-ideal so $f(a) \leq f(pqz) \leq \max\{f(p), f(z)\}$ and hence

$$(f * \mathcal{O} * f)(a) \geq \bigwedge_{(y,z)\in A_a} \bigwedge_{(p,q)\in A_y} f(a) = f(a).$$

Thus $f \preceq f * \mathcal{O} * f$. □

Lemma 3.11. [16] *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:*

(1) *S is regular.*
(2) *$B = (BSB]$ for every bi-ideal B of S.*
(3) *$B(a) = (B(a)SB(a)]$ for every $a \in S$.*

Now we characterize regular semigroups in terms of anti-fuzzy bi-ideals.

Theorem 3.12. *An ordered semigroup S is regular if and only if for every anti-fuzzy bi-ideal f of S we have $f * \mathcal{O} * f = f$.*

Proof. $\Longrightarrow$. Suppose that S is regular and f an anti-fuzzy bi-ideal of S. Let $a \in S$, since S is regular, there exists $x \in S$ such that $a \leq axa \Longrightarrow (a, xa) \in A_a$ and so $A_a \neq \varnothing$. Then

$$\begin{aligned}(f * \mathcal{O} * f)(a) &= \bigwedge_{(y,z)\in A_a} \{f(y) \vee (\mathcal{O} * f)(z)\} \\ &\leq \{f(a) \vee (\mathcal{O} * f)(xa)\} \\ &= f(a) \vee \left[\bigwedge_{(p,q)\in A_{xa}} \{\mathcal{O}(p) \vee f(q)\}\right] \\ &\leq \{f(a) \vee \{\mathcal{O}(x) \vee f(a)\}\} \\ &= \{f(a) \vee \{0 \vee f(a)\}\} \\ &= f(a) \vee f(a) \\ &= f(a)\end{aligned}$$

Hence $(f * \mathcal{O} * f)(a) \leq f(a)$. On the other hand by Proposition 11, we have

$$f(a) \leq (f * \mathcal{O} * f)(a) \text{ and it follows that } f * \mathcal{O} * f = f.$$

$\Longleftarrow$. Suppose that $f * \mathcal{O} * f = f$ for every anti-fuzzy bi-ideal f of S. Let B be a bi-ideal of S and $y \in B$. Since B is bi-ideal of S, so f_{B^C} is an anti-fuzzy bi-ideal of S, and by hypothesis

$$f_{B^C} * \mathcal{O} * f_{B^C} = f_{B^C}$$

As $y \in B$ so $y \notin B^C \implies f_{B^C}(y) = 0$ and so

$$(f_{B^C} * \mathcal{O} * f_{B^C})(y) = 0.$$

Now

$$(f_{B^C} * \mathcal{O} * f_{B^C})(y) = (f_{B^C} * f_{S^C} * f_{B^C})(y) = 0.$$

Thus, by Proposition 3.10, we have

$$(f_{B^C} * f_{S^C} * f_{B^C})(y) = f_{(BSB]^C}(y) = 0 \Longrightarrow y \notin (BSB]^C$$

and so $y \in (BSB]$. Hence we have $B \subseteq (BSB]$. On the other hand since $BSB \subseteq B$ so we have $(BSB] \subseteq (B] = B$. Thus S is regular. □

Lemma 3.13. [10] *An ordered semigroup S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.*

Theorem 3.14. *An ordered semigroup $(S, \cdot, \leq)$ is regular, left and right simple if and only if every anti-fuzzy bi-ideal is a constant mapping.*

Proof. Suppose that S is regular, left and right simple and let f be an anti-fuzzy bi-ideal of S. We consider the set $E_S := \{e \in S | e \leq e^2\}$. The set E_S is non-empty, because let $a \in S$ $(S \neq \varnothing)$, since S is regular, there exist an element $x \in S$ such that $a \leq axa$. For the element $ax \in S$, we have

$$ax \leq (axa)x = (ax)(ax) = (ax)^2 \implies ax \in E_S$$

Now let $t \in E_S (E_S \neq \varnothing)$. First we prove that f is a constant mapping over E_S i.e. for every $e \in E_S$, $f(e) = f(t)$. For this let $e \in E_S$ be an arbitrary element then $e \in S$, also we have $t \in S$ and as S is left and right simple, we have $(St] = S$, $(tS] = S$ so we get $e \in (St]$ and $e \in (tS]$. Hence there exist $y, z \in S$ such that

$$e \leq yt \text{ and } e \leq tz \implies e^2 \leq (tz)(yt) \implies e^2 \leq t(zy)t,$$

and since f is anti-fuzzy bi-ideal of S so

$$f(e^2) \leq f(t(zy)t) \leq \max\{f(t), f(t)\} = f(t) \implies f(e^2) \leq f(t)$$

Also $e \in E_S$, we have

$$e \leq e^2 \implies f(e) \leq f(e^2),$$

and we get

$$f(e) \leq f(e^2) \leq f(t) \implies f(e) \leq f(t).$$

On the other hand, $e \in S$ and S is left and right simple so $(Se] = S$, $(eS] = S$ and since $t \in S$, we get $t \in (Se]$, $t \in (eS]$ so there exist $p, q \in S$ such that

$$t \leq ep \text{ and } t \leq qe \implies t^2 \leq (ep)(qe) = e(pq)e.$$

Since f is an anti-fuzzy bi-ideal of S, we have

$$t^2 \leq e(pq)e \implies f(t^2) \leq f(e(pq)e) \leq \max\{f(e), f(e)\} = f(e)$$

but

$$t \in E_S \implies t \leq t^2 \implies f(t) \leq f(t^2) \implies f(t) \leq f(e)$$

so $f(e) = f(t)$ and hence f is a constant mapping on E_S.

Next we prove that f is a constant mapping on S. For this let $a \in S$ then as S is regular, there exists $x \in S$ such that $a \leq axa$. Now

$$ax \leq (axa)x = (ax)(ax) = (ax)^2 \implies ax \in E_S$$

And

$$xa \leq x(axa) = (xa)(xa) = (xa)^2 \implies xa \in E_S$$

Hence

$$f(ax) = f(t) \text{ and } f(xa) = f(t)$$

Again

$$a \leq axa \leq (axa)(xa) = (ax)a(xa) \text{ and we have}$$
$$f(a) \leq f((ax)a(xa)) \implies f(a) \leq \max\{f(ax), f(xa)\}.$$

So that

$$f(a) \leq \max(f(t), f(t)\} \implies f(a) \leq f(t)$$

Since S is left and right simple, we have $(Sa] = S$ and $(aS] = S$. Since $t \in S \implies t \in (Sa]$ and $t \in (aS]$. So we can find $p_1, q_1 \in S$ such that

$$t \leq p_1 a \text{ and } t \leq aq_1 \implies t^2 \leq (aq_1)(p_1 a)$$

Since f is anti-fuzzy bi-ideal so from above we get

$$f(t^2) \leq f((aq_1)(p_1 a)) \implies f(t^2) \leq f(a(q_1 p_1)a)$$

Thus

$$f(t^2) \leq \max\{f(a), (f(a)\} \implies f(t^2) \leq f(a)$$

but $t \in E_S \implies t \leq t^2$ and so $f(t) \leq f(t^2) \leq f(a)$. Hence $f(a) = f(t)$ and so f is constant mapping on S.

Conversely , let f be an anti-fuzzy bi-ideal of S and let f be a constant mapping on S. We prove that S is left, right simple and regular. For this let $a \in S$, we consider the set $(Sa]$. Now

$$(Sa](Sa] \subseteq (SaSa] \subseteq (SSa] \subseteq (Sa]$$
$$(Sa]S(Sa] = (Sa](S](Sa] \subseteq (SaS^2a] \subseteq (SaSa] \subseteq (SSa] \subseteq (Sa]$$

Let $y \in (Sa]$ and let $x \in S$ such that $x \leq y$, then $y \in (Sa] \implies$ there exists $s \in S$ such that $y \leq sa$, so $x \leq y \leq sa \implies x \leq sa$ and so $x \in (Sa]$. Hence $(Sa]$ is bi-ideal of S. So, the characteristic function $f_{(Sa]^C} : S \longrightarrow \{0, 1\}$ defined by

$$f_{(Sa]^C}(x) = \begin{cases} 1 \text{ if } x \in (Sa]^C \\ 0 \text{ if } x \notin (Sa]^C \end{cases}$$

is an anti-fuzzy bi-ideal of S. Moreover we are given that $f_{(Sa]^C}$ is a constant mapping over S. Let $(Sa] \subset S$, we consider an element $p \in S$ such that $p \notin (Sa]$, so $p \in (Sa]^C \implies f_{(Sa]^C}(p) = 1$. On the other hand, since $a^2 \in (Sa]$, so $a^2 \notin (Sa]^C \implies f_{(Sa]^C}(a^2) = 0$, which is not possible. So $(Sa] = S$. Similarly we can prove that $(aS] = S$. Hence S is left and right simple, and hence $S = (aS] = (a(Sa]] = (aSa]$, so S is regular. □

Lemma 3.15. [16] *Let S be an ordered semigroup, then the following are equivalent.*
(1) *S is both regular and intra-regular.*
(2) *$B = (B^2]$ for every bi-ideal B of S*
(3) *$B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$ for all bi-ideals B_1 and B_2 of S.*

Lemma 3.16. [17]. *Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is an anti-fuzzy subsemigroup if and only if $f * f \succeq f$.*

Corollary 3.17. *Let S be an ordered semigroup and f be anti-fuzzy bi-ideal of S then $f \preceq f * f$.*

Proposition 3.18. *If f and g are anti-fuzzy bi-ideals of an ordered semigroup S then $f \vee g$ is also an anti-fuzzy bi-ideal.*

Proof. Let f, g be anti-fuzzy bi-ideals of S. Obviously $f \vee g$ is an anti-fuzzy subsemigroup. Let $x, y, z \in S$ then

$$\begin{aligned}(f \vee g)(x) \vee (f \vee g)(z) &= \{f(x) \vee g(x) \vee f(z) \vee g(z)\} \\ &= [\{f(x) \vee f(z)\} \vee \{g(x) \vee g(z)\}]\end{aligned}$$

Since f, g are anti-fuzzy bi-ideals so

$$f(xyz) \leq \{f(x) \vee f(z)\} \text{ and } g(xyz) \leq \{g(x) \vee g(z)\}$$

So from above we conclude that

$$\begin{aligned}(f \vee g)(x) \vee (f \vee g)(z) &\geq f(xyz) \vee g(xyz) \\ &= (f \vee g)(xyz)\end{aligned}$$

Thus $(f \vee g)(xyz) \leq (f \vee g)(x) \vee (f \vee g)(z)$. Let $x, y \in S$ such that $x \leq y$ then

$$\begin{aligned}f(x) &\leq f(y) \text{ and } g(x) \leq g(y) \text{ because } f \text{ and } g \text{ are anti-fuzzy bi-ideals} \\ \Longrightarrow & \ f(x) \vee g(x) \leq f(y) \vee g(y)\end{aligned}$$

Hence $(f \vee g)(x) \leq (f \vee g)(y)$. □

Proposition 3.19. *If f and g are fuzzy subsets of ordered semigroup S then $f * g \preceq (f \vee g) * (f \vee g)$*

Proof. Let $a \in S$. If $A_a = \varnothing$ then $((f \vee g) * (f \vee g))(a) = 1 \geq (f * g)(a)$. If $A_a \neq \varnothing$ then

$$\begin{aligned}((f \vee g) * (f \vee g))(a) &= \bigwedge_{(y,z) \in A_a} \{(f \vee g)(y) \vee (f \vee g)(z)\} \\ &= \bigwedge_{(y,z) \in A_a} \{f(y) \vee g(y) \vee f(z) \vee g(z)\} \\ &\geq \bigwedge_{(y,z) \in A_a} \{f(y) \vee g(z)\} \\ &= (f * g)(a)\end{aligned}$$

Hence $f * g \preceq (f \vee g) * (f \vee g)$. □

Lemma 3.20. *Let S be an ordered semigroup, f and g be anti-fuzzy bi-ideals of S then the product $f * g$ is also an anti-fuzzy bi-ideal of S.*

Proof. Suppose S is an ordered semigroup and f, g are anti-fuzzy bi-ideals of S. Let $a \in S$. If $A_a = \varnothing$ then

$$((f * g) * (f * g))(a) = 1 \geq (f * g)(a)$$

If $A_a \neq \varnothing$ Then

$$\begin{aligned}
& ((f * g) * (f * g))(a) \\
= & \bigwedge_{(y,z)\in A_a} \{(f * g)(y) \vee (f * g)(z)\} \\
= & \bigwedge_{(y,z)\in A_a} \left[\left[\bigwedge_{(p_1,q_1)\in A_y} \{f(p_1) \vee g(q_1)\}\right] \vee \left[\bigwedge_{(p_2,q_2)\in A_z} \{f(p_2) \vee g(q_2)\}\right]\right] \\
= & \bigwedge_{(y,z)\in A_a} \bigwedge_{(p_1,q_1)\in A_y} \bigwedge_{(p_2,q_2)\in A_z} \{f(p_1) \vee g(q_1) \vee f(p_2) \vee g(q_2)\} \\
= & \bigwedge_{(y,z)\in A_a} \bigwedge_{(p_1,q_1)\in A_y} \bigwedge_{(p_2,q_2)\in A_z} \{\{f(p_1) \vee f(p_2) \vee g(q_1)\} \vee g(q_2)\} \\
\geq & \bigwedge_{(y,z)\in A_a} \bigwedge_{(p_1,q_1)\in A_y} \bigwedge_{(p_2,q_2)\in A_z} \{\{f(p_1) \vee f(p_2)\} \vee g(q_2)\}
\end{aligned}$$

As $(y, z) \in A_a \implies a \leq yz$, $(p_1, q_1) \in A_y \implies y \leq p_1q_1$, $(p_2, q_2) \in A_z \implies z \leq p_2q_2$. So we have $a \leq yz \leq p_1q_1z \leq p_1q_1p_2q_2 = (p_1q_1p_2)q_2$ and $(p_1q_1p_2, q_2) \in A_a$. Now

$$\begin{aligned}
& \bigwedge_{(y,z)\in A_a} \bigwedge_{(p_1,q_1)\in A_y} \bigwedge_{(p_2,q_2)\in A_z} \{\{f(p_1) \vee f(p_2)\} \vee g(q_2)\} \\
\geq & \bigwedge_{(p_1q_1p_2,q_2)\in A_a} \{\{f(p_1) \vee f(p_2)\} \vee g(q_2)\}.
\end{aligned}$$

Since f is anti-fuzzy bi-ideal so $f(p_1q_1p_2) \leq \{f(p_1) \vee f(p_2)\}$. Hence

$$\begin{aligned}
\bigwedge_{(p_1q_1p_2,q_2)\in A_a} \{\{f(p_1) \vee f(p_2)\} \vee g(q_2)\} \geq & \bigwedge_{(p_1q_1p_2,q_2)\in A_a} \{f(p_1q_1p_2) \vee g(q_2)\} \\
= & \bigwedge_{(p,q)\in A_a} \{f(p) \vee g(q)\} \\
= & (f * g)(a)
\end{aligned}$$

Thus$((f*g)*(f*g))(a) \geq (f*g)(a)$. Let $x, y, z \in S$ then

$$\begin{aligned}
(f*g)(x) \vee (f*g)(z) &= \left[\bigwedge_{(a,b)\in A_x} \{f(a) \vee g(b)\}\right] \vee \left[\bigwedge_{(c,d)\in A_z} \{f(c) \vee g(d)\}\right] \\
&= \bigwedge_{(a,b)\in A_x} \bigwedge_{(c,d)\in A_z} \{f(a) \vee g(b) \vee f(c) \vee g(d)\} \\
&= \bigwedge_{(a,b)\in A_x} \bigwedge_{(c,d)\in A_z} [\{f(a) \vee f(c)\} \vee \{g(b) \vee g(d)\}] \\
&= \bigwedge_{(a,b)\in A_x} \bigwedge_{(c,d)\in A_z} [\{f(a) \vee f(c) \vee g(b)\} \vee g(d)] \\
&\geq \bigwedge_{(a,b)\in A_x} \bigwedge_{(c,d)\in A_z} [\{f(a) \vee f(c)\} \vee g(d)]
\end{aligned}$$

As $x \leq ab$ and $z \leq cd$ so $xyz \leq (ab)y(cd) = (a(by)c)d \Longrightarrow (a(by)c, d) \in A_{xyz}$. Thus

$$\bigwedge_{(a,b)\in A_x} \bigwedge_{(c,d)\in A_z} [\{f(a) \vee f(c)\} \vee g(d)] = \bigwedge_{(a(by)c,d)\in A_{xyz}} [\{f(a) \vee f(c)\} \vee g(d)]$$

Since f is an anti-fuzzy bi-ideal so $f(a(by)c) \leq \{f(a) \vee f(c)\}$. Hence

$$\begin{aligned}
\bigwedge_{(a(by)c,d)\in A_{xyz}} [\{f(a) \vee f(c)\} \vee g(d)] &\geq \bigwedge_{(a(by)c,d)\in A_{xyz}} \{f(a(by)c) \vee g(d)\} \\
&\geq \bigwedge_{(p_3,q_3)\in A_{xyz}} \{f(p_3) \vee g(q_3)\} \\
&= (f*g)(xyz)
\end{aligned}$$

Thus $(f*g)(xyz) \leq (f*g)(x) \vee (f*g)(z)$. Let $x, y \in S$ such that $x \leq y$. If $(p,q) \in A_y$ then

$$y \leq pq \implies x \leq y \leq pq \implies x \leq pq.$$

Hence $A_y \subseteq A_x$. If $A_x = \varnothing$ then $A_y = \varnothing$ and so $(f*g)(x) = 1 = (f*g)(y)$. If $A_y \neq \varnothing$ then $A_x \neq \varnothing$, so

$$\begin{aligned}
(f*g)(y) &= \bigwedge_{(p,q)\in A_y} \{f(p) \vee g(q)\} \\
&\geq \bigwedge_{(c,d)\in A_x} \{f(c) \vee g(d)\} \\
&= (f*g)(x)
\end{aligned}$$

Hence $(f*g)(x) \leq (f*g)(y)$. Consequently $f*g$ is an anti-fuzzy bi-ideal of S. □

Lemma 3.21. *Let S be an ordered semigroup, f and g be fuzzy subsets of S. Then $f * \mathcal{O} \preceq f * g$(resp. $\mathcal{O} * g \preceq f * g$).*

Proof. Let $a \in S$. If $A_a = \varnothing$ then $(f*g)(a) = 1 \geq (f*\mathcal{O})(a)$. Let $A_a \neq \varnothing$ then

$$(f*g)(a) = \bigwedge_{(p,q)\in A_a} \{f(p) \vee g(q)\}$$

As $g(q) \geq \mathcal{O}(q)$ for all $q \in S$. Thus

$$(f * g)(a) = \bigwedge_{(p,q)\in A_a} \{f(p) \vee g(q)\} \geq \bigwedge_{(p,q)\in A_a} \{f(p) \vee \mathcal{O}(q)\} = (f * \mathcal{O})(a)$$

Hence $f * \mathcal{O} \preceq f * g$. Similarly, we can prove that $\mathcal{O} * g \preceq f * g$. □

In the following Theorem we characterize regular and intra-regular ordered semigroup in terms of anti-fuzzy bi-ideals.

Theorem 3.22. *Let S be an ordered semigroup. Then the following are equivalent:*

(1) *S is both regular and intra-regular.*
(2) *$f * f = f$ for every anti-fuzzy bi-ideal f of S.*
(3) *$f \vee g = f * g \vee f * g$ for all anti-fuzzy bi-ideals f and g of S.*

Proof. (1) $\Longrightarrow$ (2). Suppose that S is both regular and intra-regular and f an anti-fuzzy bi-ideal of S. Since S is both regular and intra-regular so there exist $x, y, z \in S$ such that $a \leq axa$ and $a \leq ya^2z$. Thus

$$a \leq axa \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa)$$

Hence $(axya, azxa) \in A_a$ so $A_a \neq \phi$. Thus

$$\begin{aligned}(f * f)(a) &= \bigwedge_{(y,z)\in A_a} \{f(y) \vee f(z)\} \\ &\leq \{f(axya) \vee f(azxa)\}\end{aligned}$$

Since f is anti-fuzzy bi-ideal of S so we have

$$f(axya) \leq f(a) \vee f(a) = f(a)$$

and

$$f(azxa) \leq f(a) \vee f(a) = f(a)$$

So that

$$\begin{aligned}(f * f)(a) &\leq \{f(axya) \vee f(azxa)\} \\ &\leq f(a) \vee f(a) = f(a)\end{aligned}$$

Hence $f * f \preceq f$. By Corollary 3.17, we have $f \preceq f * f$. Hence $f = f * f$.

(2) $\Longrightarrow$ (3) Let f and g be anti-fuzzy bi-ideals of S then by Proposition 3.18, $f \vee g$ is also an anti-fuzzy bi-ideal of S. By (2),

$$f \vee g = (f \vee g) * (f \vee g)$$

But by Proposition 3.19, $f * g \preceq (f \vee g) * (f \vee g)$. Thus

$$f * g \preceq (f \vee g) * (f \vee g) = f \vee g$$

Similarly, we can prove that $g * f \preceq f \vee g$. Thus $(f * g) \vee (g * f) \preceq f \vee g$.

On the other hand, by Lemma 3.20, $f * g$ and $g * f$ are anti-fuzzy bi-ideals of S and so $(f * g) \vee (g * f)$ is an anti-fuzzy bi-ideal. By (2) we have

$$\begin{aligned}(f * g) \vee (g * f) &= ((f * g) \vee (g * f)) * ((f * g) \vee (g * f)) \\ &\succeq (f * g) * (g * f) \text{ By Proposition 3.19} \\ &= f * (g * g) * f \\ &= f * g * f \text{ (as } g * g = g \text{ by (2) above)} \\ &\succeq f * \mathcal{O} * f \text{ (as } f * g \succeq f * \mathcal{O} \text{ by Lemma 3.21)} \\ &\succeq f \text{ (by Proposition 3.19)}\end{aligned}$$

Hence $f \preceq (f * g) \vee (g * f)$. Similarly we can prove that $g \preceq (f * g) \vee (g * f)$. Thus $f \vee g \preceq (f * g) \vee (g * f)$. Therefore $f \vee g = (f * g) \vee (g * f)$.

(3) $\Longrightarrow$ (1). Let $f \vee g = (f * g) \vee (g * f)$ for every anti-fuzzy bi-ideal f and g of S. By Lemma 3.16 it is enough to prove that

$$B_1 \cap B_2 = (B_1 B_2] \cap (B_2 B_1] \text{ for all bi-ideals } B_1 \text{ and } B_2 \text{ of } S.$$

Let $y \in B_1 \cap B_2$ then $y \in B_1$ and $y \in B_2$, then $f_{B_1^C}$ and $f_{B_2^C}$ are anti-fuzzy bi-ideals of S. By hypothesis,

$$(f_{B_1^C} \vee f_{B_2^C})(y) = (f_{B_1^C} * f_{B_2^C} \vee f_{B_2^C} * f_{B_1^C})(y)$$

As $y \in B_1$ and $y \in B_2 \Longrightarrow y \notin B_1^C$ and $y \notin B_2^C$, therefore $f_{B_1^C}(y) = 0$ and $f_{B_2^C}(y) = 0$. Thus $(f_{B_1^C} \vee f_{B_2^C})(y) = 0$. Hence $(f_{B_1^C} * f_{B_2^C} \vee f_{B_2^C} * f_{B_1^C})(y) = 0$. By Proposition 15, we have $f_{B_1^C} * f_{B_2^C} = f_{(B_1B_2]^C}$ and $f_{B_2^C} * f_{B_1^C} = f_{(B_2B_1]^C}$. Hence

$$f_{(B_1B_2]^C} \vee f_{(B_2B_1]^C} = f_{(B_1B_2]^C \cup (B_2B_1]^C} = f_{((B_1B_2] \cap (B_2B_1])^C}.$$

Thus, we have

$$f_{((B_1B_2] \cap (B_2B_1])^C}(y) = 0 \Longrightarrow y \notin ((B_1B_2] \cap (B_2B_1])^C.$$

Hence $y \in (B_1B_2] \cap (B_2B_1]$. Thus

$$B_1 \cap B_2 \subseteq (B_1B_2] \cap (B_2B_1].$$

On the other hand, let $y \in (B_1B_2] \cap (B_2B_1]$, then $y \notin ((B_1B_2] \cap (B_2B_1])^C$, so $f_{((B_1B_2] \cap (B_2B_1])^C} = 0$. Now

$$\begin{aligned}0 &= f_{((B_1B_2] \cap (B_2B_1])^C}(y) \\ &= f_{(B_1B_2]^C \cup (B_2B_1]^C}(y) \\ &= (f_{(B_1B_2]^C} \vee f_{(B_2B_1]^C})(y) \\ &= (f_{B_1^C} * f_{B_2^C}) \vee (f_{B_2^C} * f_{B_1^C})(y)\end{aligned}$$

Since $(f_{B_1^C} * f_{B_2^C}) \vee (f_{B_2^C} * f_{B_1^C}) \succeq f_{B_1^C}$ and $(f_{B_1^C} * f_{B_2^C}) \vee (f_{B_2^C} * f_{B_1^C}) \succeq f_{B_2^C}$, so

$$0 = (f_{B_1^C} * f_{B_2^C}) \vee (f_{B_2^C} * f_{B_1^C})(y) \succeq (f_{B_1^C} \vee f_{B_2^C})(y)$$

Thus

$$\begin{aligned}(f_{B_1^C} \vee f_{B_2^C})(y) &= 0 \Longrightarrow f_{B_1^C \cup B_2^C}(y) = 0 \Longrightarrow f_{(B_1 \cap B_2)^C} = 0 \\ &\Longrightarrow y \notin (B_1 \cap B_2)^C \Longrightarrow y \in B_1 \cap B_2\end{aligned}$$

Hence

$$(B_1B_2] \cap (B_2B_1] \subseteq B_1 \cap B_2. \text{ Therefore } (B_1B_2] \cap (B_2B_1] = B_1 \cap B_2.$$

Consequently S is both regular and intra-regular. □

Lemma 3.23. [5] *An ordered semigroups S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$ or equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.*

We denote by $B(a) = (a \cup a^2 \cup aSa]$, the bi-ideal of S generated by a $(a \in S)$. If $A \subseteq S$ then $B(A) = (A \cup A^2 \cup ASA]$ is the smallest bi-ideal generated by A.

Theorem 3.24. *An ordered semigroup $(S, \cdot, \leq)$ is completely regular if and only if for every anti-fuzzy bi-ideal f of S, we have $f(a) = f(a^2)$ for every $a \in S$.*

Proof. Suppose that f is an anti-fuzzy bi-ideal of SSince S is completely regular so for every $a \in S$ there exists $x \in S$ such that $a \leq a^2xa^2$. Since f is anti-fuzzy bi-ideal, we have

$$f(a) \leq f(a^2xa^2) \implies f(a) \leq f(a^2) \vee f(a^2) = f(a^2) \leq f(a) \vee f(a) = f(a)$$

Hence $f(a) = f(a^2)$.

Conversely, we consider the bi-ideal of S generated by a^2, that is

$$B(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$$

Since $B(a^2)$ is bi-ideal of S so $f_{B(a^2)^C}$ is an anti-fuzzy bi-ideal of S and hence by hypothesis $f_{B(a^2)^C}(a) = f_{B(a^2)^C}(a^2)$ for every $a \in S$. Since $a^2 \in B(a^2)$ so $a^2 \notin B(a^2)^C \implies f_{B(a^2)^C}(a^2) = 0$ and hence

$$f_{B(a^2)^C}(a) = 0 \implies a \notin B(a^2)^C \implies a \in B(a^2).$$

Thus $a \in (a^2 \cup a^4 \cup a^2Sa^2]$, and $a \leq y$ for some $y \in (a^2 \cup a^4 \cup a^2Sa^2]$. If $y = a^2$ then

$$a \leq y = a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$$

Which shows that $a \in a^2Sa^2$ and hence $a \in (a^2Sa^2]$. If $y = a^4$ then

$$a \leq y = a^4 = aaa^2 \leq a^4aa^2 = a^2(a^2a)a^2 = a^2a^3a^2 \in a^2Sa^2$$

And again we obtain $a \in a^2Sa^2 \implies a \in (a^2Sa^2]$. If $y \in a^2Sa^2$ then

$$a \leq y \in a^2Sa^2 \implies a \in a^2Sa^2$$

and so $a \in (a^2Sa^2]$. Consequently S is completely regular. □

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